AN INVESTIGATION OF THE CONCEPTS OF PURPOSEFUL SEARCHES

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THESIS

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ABSTRACT

A discrete-time discrete-space search model is considered in which an observer employing an idealized detection device is searching for a uniformly distributed stationary target. The model is formulated as a discrete-time counting process, called the search process, which under weak additional conditions is uniquely determined by a sequence of probabilities. Formulas for the time-to-detection and the detection rate of a search are derived in terms of the parameters of the search process, and are applied to two special types of searches, the systematic search and the random search. Using these search types as boundary cases a purposeful search is defined, and sufficient conditions on the sequence of probabilities are established for the purposeful search. Possible extensions of the search process to less restricted models are indicated.

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1. INTRODUCTION

Systematic research in the field now commonly known as search theory was begun during World War II when the Anti-Submarine Warfare Operations Research Group (ASWORG) of the U. S. Navy studied the German submarine threat to allied shipping and developed methods and tactics to counter this threat. The major work done by this organization is covered in Koopman's book Search and Screening [Ref. 1] which, although now more than 25 years old, is still the fundamental work on the subject.

In the years since then the field of search has grown and expanded. A number of specific problems have been formulated and solved, new areas of application have been added (e.g., mineralogical surveying, commercial fishing), new methodologies have been applied (e.g., game theory, decision analysis), and researchers from other disciplines, such as electrical engineers, have become involved. Enslow's bibliography with abstracts [Ref. 2] and Dobbie's [Ref. 3] and Pollock's [Ref. 4] articles provide an overview of this growth. Despite this, "the field still appears to be relatively unstructured" and "very little general theory has evolved" as Enslow[1] remarks.

The present paper approaches search theory from a point of view which apparently has not been taken before.

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[1]Enslow [Ref. 2], p. 177

In doing so, the aim is not to solve another specific problem, i.e., to find an optimal search method for some particular context, but rather to describe the basic and essential structure of all "meaningful" or "purposeful" searches in the hope that this may be a way to get closer to a unifying theory of search.

2. THE SEARCH MODEL

Throughout this paper the following search model is considered:

> One observer is searching for one stationary target which is equally likely to be in any one of n squares which constitute the search area. At each time step the observer searches one square; if the target is in that square it is detected and the search ends; if the target is not in that square the observer knows that it is not in that square and goes on to search another (possibly the same) square. The observer is not constrained to search the squares in any particular order.

The model assumes that the observer uses a definite range law detection device[2] which has zero probability of false alarms[3]. This assumption implies that any observer who wants to detect the target would never want to search the same square more than once. It is not unreasonable, however, to assume that, unwillingly and unknowingly, he may search squares more than once

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[2]A definite range law detection device, commonly called a "cookie cutter" device, detects a target within its range with probability one.

[3]A false alarm is the report of a detection when in fact the target is not present.

due, for instance, to navigational errors, limited memory, etc. Hence the actual path of the observer through the search area is determined by chance. Of course, when the observer's detection device is not of the definite range law type, searching a square more than once may be a sensible course of action.

Clearly the model deviates very much from the reality of traditional naval searches such as a raider searching for merchant ships, destroyers hunting a submarine, or SAR (search and rescue) units trying to locate the crew of a downed aircraft. As far as modern methods of anti-submarine warfare are concerned (e.g. those which employ ASW helicopters with sono buoys) it may be a reasonable first approximation; and there are situations which it describes rather accurately; consider, for instance, a student trying to find a formula which he knows is written on one of the many pieces of paper scattered across his desk.

3. MATHEMATICAL FORMULATION OF THE SEARCH MODEL

The model under consideration can be described mathematically in a number of ways. Since the position of the target and in most cases also the path of the observer are chance dependent, any description will necessarily be probabilistic (rather than analytic) in nature. A discrete-time counting process has been chosen here because it is felt that this particular formulation has intuitive appeal and exposes the essential features of a search quite well. It will be developed in three stages.

3.1 THE AREA ACQUISITION PROCESS

Suppose for the moment that, unknown to the observer, the target is not present in the search area, and assume that the search begins with time step one. Define $A(0) \equiv 0$, and for $m = 1,2...$ let $A(m)$ be the number of different squares which the observer has searched by time step m. Since at each time step the observer searches exactly one square, one square is searched by time step one and hence $A(1) = 1$. At each successive time step the square searched may be either one which was not searched previously (a "new" square) or else one which was searched previously (a "used" square), until all n squares have been searched. Then only used squares can be searched. Thus $A(m)$, $m = 1,2,...,$ are random variables which take on the values $1,2,..., \min\{m,n\}$.

The sequence $\langle A(m),\ m = 0,1,\ldots\rangle$ will be called the <u>area acquisition process</u> and the search of a new square an <u>acquisition event</u>. The $k^{\underline{th}}$ <u>event time</u>, $S(k)$, is the number of the time step at which the $k^{\underline{th}}$ acquisition event occurs, i.e.

$$S(k) \equiv \min\{m \mid A(m) = k\},\ k = 1,\ldots,n+1.$$

The <u>inter-event times</u> $T(k)$ are defined by

$$T(1) \equiv S(1)$$

$$T(k) \equiv S(k)-S(k-1),\ k = 2,\ldots,n+1.$$

Since $A(1) = 1$, thus $S(1) = 1$ and $T(1) = 1$; also, since an $(n+1)^{\underline{st}}$ acquisition event can never happen (recall that there are only n squares), $S(n+1)$ and $T(n+1)$ are both infinite.

In the acquisition process the fundamental relationship between the number of events and the event times in a counting process takes on the form

$$\{S(k) \leq m\} \text{ iff } \{A(m) \geq k\}.$$

To derive the distributions of the inter-event times consider the random variables $X(m)$ defined by

$$X(m) = \begin{cases} 1, & \text{if an acquisition event occurs at time step } m \\ 0, & \text{otherwise} \end{cases}$$

for all positive integers m.

For most practical searches one would expect the probability of searching a new square at time step m to be a function of the number of different squares searched before time step m (the greater the number of squares searched the fewer new squares are left and the more likely an old square is searched again), but not a function

of the time step itself. Obviously, exceptions are possible (the crew of the observer may get tired and make navigational errors after some time, no matter how many different squares have been searched). Moreover, other factors may also influence this probability: the weather, the time of day, the observer's navigator, etc. These factors, however, are neglected here, and it is postulated that the probability is a function only of the number of different squares searched so far, i.e.

$$P[X(m) = 1 | A(m-1) = k] = \alpha(k),\ m = 1,2,\ldots,$$

where $\alpha(k)$ is a parameter which depends on k only. Clearly,

$$\alpha(0) = P[X(1) = 1 | A(0) = 0] = 1$$

since the first square searched is always a new square, and

$$\alpha(n) = P[X(m) = 1 | A(m-1) = n] = 0$$

because no new square can be searched after n squares have been searched. For intermediate values of k

$$0 \leq \alpha(k) \leq 1,$$

since the $\alpha(k)$'s are probabilities.

The distribution of the inter-event times can now be derived; for any $k = 1,\ldots,n+1$ and any $j = 1,2,\ldots,$ $P[T(k) > j]$ is the probability that at all of the j time steps immediately following the time step at which the $(k-1)^{\underline{st}}$ event has occurred no event will occur. By the definition of $\alpha(k)$, however, there is probability $[1-\alpha(k-1)]$ that no event will occur at each of the j time

steps; hence $P[T(k) > j] = [1-\alpha(k-1)]^j$. Thus it follows that $T(1),\ldots,T(n+1)$ are independent and that $T(k)$ has the geometric distribution with success probability $p = \alpha(k-1)$. In particular, $T(1) = 1$ and $T(n+1) = +\infty$, which agrees with previous results.

3.2 THE TARGET VARIABLE

The target and its detection which have been neglected so far are now taken into account by introducing the detection event into the area acquisition process. This event is bound to occur simultaneously with an acquisition event because the target can be detected only when a new square is searched, and is equally likely to occur at any one of the n acquisition events since the target is uniformly distributed over the n squares. Stated formally: the detection event occurs at the $Z^{\underline{th}}$ acquisition event where the random variable Z, called the target variable, is independent of the acquisition process and has probability mass function $P[Z = k] = \frac{1}{n}$, $k = 1,\ldots,n$.

3.3 THE SEARCH PROCESS

The area acquisition process and the target variable together form the search process. Its explicit definition is given here as a summary of the results of this section.

The search process is a pair $\{\langle A(m),\ m = 0,1,\ldots\rangle, Z\}$ where

(1) the area acquisition process $\langle A(m),\ m = 0,1,\ldots\rangle$

counts the number of acquisition events in any interval $(0,m]$; its inter-event times $T(k)$, $k = 1,\ldots,n+1$, are such that

(a) $T(1),\ldots,T(n+1)$ are independent, and

(b) $T(k)$ has the geometric distribution with success probability $p = \alpha(k-1)$ where

(c) the parameters $\alpha(k)$, $k = 0,\ldots,n$, satisfy the relation $0 = \alpha(n) \leq \alpha(k) \leq \alpha(0) = 1$;

(2) the target variable Z with probability mass function $P[Z = k] = \frac{1}{n}$, $k = 1,\ldots,n$, is independent of the acquisition process; and which

(3) terminates with the detection at the time of the occurrence of the $Z^{\underline{th}}$ acquisition event.

The search process constitutes the mathematical formulation of the proposed search model under the assumption that the probability of searching a new square depends only on the number of different squares searched so far. It is important to note, and obvious from the definition, that the search process is uniquely specified by a sequence $\langle\alpha(k),\ k = 0,\ldots,n\rangle$.

4. TIME-TO-DETECTION AND DETECTION RATE

Two quantities of particular interest in a search are the time-to-detection and the detection rate.

The time-to-detection, D, is the number of the time step at which the detection occurs; in terms of the search process it is the number of the time step at which the $Z^{\underline{th}}$ acquisition event occurs, i.e.

$$D = S(Z).$$

Its distribution function, denoted by F_D, can be expressed as follows:

$$\begin{aligned}
F_D(m) &\equiv P[D \leq m] \\
&= P[S(Z) \leq m] \\
&= \sum_{k=1}^{n} P[S(Z) \leq m | Z = k] \cdot P[Z = k] \\
&= \sum_{k=1}^{n} P[S(k) \leq m] \cdot P[Z = k] \\
&= \sum_{k=1}^{n} P[A(m) \geq k] \cdot P[Z = k] \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n} P[A(m) \geq k] \\
&= \frac{1}{n} \cdot EA(m). \qquad (1)
\end{aligned}$$

In a discrete-time model the detection rate, d(m), is defined to be the probability that detection occurs at time step m given it has not occurred before,

$$d(m) \equiv P[D = m | D > m-1],$$

for all integers m such that $P[D > m-1] > 0$[4]. $d(m)$ can be written in terms of F_D as

$$d(m) = P[D = m|D > m-1]$$

$$= \frac{P[D = m]}{P[D > m-1]}$$

$$= \frac{F_D(m)-F_D(m-1)}{1-F_D(m-1)} ,$$

or, using equation (1), as

$$d(m) = \frac{EA(m)-EA(m-1)}{n-EA(m-1)} . \qquad (2)$$

Consider now the quantity $EA(m)$. By definition, $A(m)$ is the number of different squares searched by time step m which is the same as the number of time steps up to and including m at which a new square is searched; hence

$$A(m) = \sum_{i=1}^{m} X(i) \qquad (3)$$

and

$$EA(m) = E \sum_{i=1}^{m} X(i)$$

$$= \sum_{i=1}^{m} EX(i)$$

$$= \sum_{i=1}^{m} P[X(i) = 1], \qquad (4)$$

since $X(i)$ is a Bernoulli random variable.

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[4]This condition is always satisfied unless $\alpha(k) = 1$ for all $k = 1,\ldots,n-1$.

$P[X(i) = 1]$ can be expressed using the definition of $\alpha(k)$ as

$$P[X(i) = 1] = \sum_k P[X(i)=1|A(i-1)=k] \cdot P[A(i-1)=k]$$

$$= \sum_k \alpha(k) \cdot P[A(i-1) = k]$$

$$= E\alpha[A(i-1)],$$

which when combined with equation (4) yields

$$EA(m) = \sum_{i=1}^{m} E\alpha[A(i-1)], \quad m = 1,2,\ldots \quad . \tag{5}$$

Equation (5) can now be used to write equation (2) in a different way. Since

$$EA(m)-EA(m-1) = \sum_{i=1}^{m} E\alpha[A(i-1)] - \sum_{i=1}^{m-1} E\alpha[A(i-1)]$$

$$= E\alpha[A(m-1)],$$

thus

$$d(m) = \frac{E\alpha[A(m-1)]}{n-EA(m-1)}, \quad \begin{matrix} m = 1,2,\ldots, \\ n > EA(m-1). \end{matrix} \tag{6}$$

Finally the distribution of the time-to-detection can be expressed in terms of the detection rate. From

$$P[D > m] = P[D \neq m, D > m-1]$$

$$= P[D \neq m|D > m-1] \cdot P[D > m-1]$$

$$= \{1-P[D = m|D > m-1]\} \cdot P[D > m-1]$$

$$= [1-d(m)] \cdot P[D > m-1]$$

$$= [1-d(m)] \cdot [1-d(m-1)] \cdot P[D > m-2]$$

$$\vdots$$

$$P[D > m] = [1-d(m)] \cdot \ldots \cdot [1-d(1)] \cdot P[D > 0]$$

$$= \prod_{i=1}^{m} [1-d(i)]$$

it follows that

$$F_D(m) = 1 - \prod_{i=1}^{m} [1-d(i)] \quad . \tag{7}$$

5. SYSTEMATIC SEARCH AND RANDOM SEARCH

At this stage it seems appropriate to apply the mathematical tools developed so far, and to discuss two very simple and well-known types of searches: one which will be called the "systematic" search, and another which is commonly known as "random" search.

In the systematic search[5] the observer is assumed to move in such a way that no part of the area is searched more than once. In terms of the model this means that at every time step a new square is searched with probability one. Hence, the search process is specified by

$$\alpha_{ss}(k) = \begin{cases} 1, & k = 0,\dots,n-1 \\ 0, & k = n. \end{cases} \tag{8}$$

From equation (8) it is immediately obvious that the A(m)'s are degenerate random variables and that

$$EA_{ss}(m) = \begin{cases} m, & m = 0,\dots,n \\ n, & m = n,n+1,\dots \end{cases} \quad . \tag{9}$$

The distribution of the time-to-detection, D_{ss}, is obtained from equations (1) and (9) as

$$\begin{aligned} F_{ss}(m) &\equiv P[D_{ss} \le m] \\ &= \begin{cases} 0, & m \le 0 \\ \frac{m}{n}, & m = 1,\dots,n \\ 1, & m > n, \end{cases} \end{aligned} \tag{10}$$

[5] Koopman [Ref. 3] considers this search as the first case in the search by parallel sweeps, but does not give it a name.

and the detection rate, $d_{ss}(m)$, from equations (2) and (9) as

$$d_{ss}(m) \equiv P[D_{ss} = m | D_{ss} > m-1]$$

$$= \frac{1}{n-m+1}, \quad m = 1,\ldots,n. \tag{11}$$

The random search assumes that the observer is equally likely to search any one of the n squares at every time step, no matter whether he has searched that particular square before or not. Thus the probability of searching a new square at any one time step is the ratio of the number of squares not searched so far and the total number of squares. But the number of squares not searched so far is just the total number of squares minus the number of squares searched so far, and thus

$$P[X_{rs}(m) = 1 | A_{rs}(m-1) = k] = \frac{n-k}{n} .$$

Therefore the sequence which specifies the search process for the random search is

$$\alpha(k) = \frac{n-k}{n}, \quad k = 0,\ldots,n. \tag{12}$$

Here it is convenient to derive the detection rate first. From equations (6) and (12) one obtains

$$d_{rs}(m) \equiv P[D_{rs} = m | D_{rs} > m-1]$$

$$= \frac{E\alpha_{rs}[A_{rs}(m-1)]}{n-EA_{rs}(m-1)}$$

$$= \frac{E[\frac{n-A_{rs}(m-1)}{n}]}{n-EA_{rs}(m-1)}$$

$$= \frac{\frac{1}{n} E[n-EA_{rs}(m-1)]}{n-EA_{rs}(m-1)}$$

$$= \frac{1}{n}, \quad m = 1,2,\ldots \quad . \tag{13}$$

This result together with equation (7) can be used to get the distribution of the time-to-detection as

$$F_{rs}(m) \equiv P[D_{rs} \leq m]$$

$$= 1- \prod_{i=1}^{m} [1-\frac{1}{n}]$$

$$= \begin{cases} 0, & m \leq 0 \\ 1-(\frac{n-1}{n})^{m}, & m = 1,2,\ldots \end{cases} \quad . \tag{14}$$

Of course, the above result could have been obtained more easily from the following direct argument: at any time step the probability of not detecting the target is $\frac{n-1}{n}$; hence the probability of not detecting the target at the first m time steps is $(\frac{n-1}{n})^m$, and thus D_{rs} has the geometric distribution with success probability $p = \frac{1}{n}$ which is also the detection rate.

Clearly the systematic search is optimal for the model, but it is equally obvious that such a search "without overlap" can be performed only under ideal conditions. It requires no less than that the observer have perfect knowledge of his position, perfect recollection of his path, and perfect ability to move wherever he wants, at each time step of the search. The random search, on the other hand, can be regarded as minimal for the model, in that an observer with the desire to

detect the target can hardly do worse. The significance, then, of the systematic and the random searches is not that they may be accurate descriptions of real world searches, but rather the fact that they can serve as boundary cases for certain types of searches which, for lack of a better term, will be called "purposeful" searches.

6. PURPOSEFUL SEARCHES

A purposeful search can be defined vaguely as a search which lies somewhere in between a random search and a systematic search, or, in other words, as a search which is at least as "good" as a random search (and not "better" than a systematic search; this condition, however, is satisfied by all possible searches in the context of the model). To make the definition precise some criteria for "good" have to be chosen. The choice, of course, is arbitrary as long as the criteria are stated in terms of quantities which are of significance in a search.

One such quantity is the time-to-detection; a "short" time-to-detection is "good", and hence a search could be called purposeful if its time-to-detection is smaller than that of a random search or, alternatively, if its expected time-to-detection is less than that of a random search. Another important quantity is the detection rate. Equations (11) and (13) show that the systematic search has a strictly increasing detection rate, whereas that of the random search is constant. This suggests as a condition for a purposeful search that its detection rate be non-decreasing. Intuitively this means that the more time the observer has spent searching for the target without finding it, the more likely he is to detect it at the next step.

In this paper a search will be called a purposeful

search if and only if

(1) its time-to-detection is not greater than the time-to-detection in the random search, and

(2) its detection rate is non-decreasing.

In subsection 3.3 it was shown that a sequence $\langle \alpha(k), k = 0,\ldots,n \rangle$ uniquely specifies a search process; hence the conditions above must be equivalent to some conditions on the $\alpha(k)$ sequence.

Sufficient conditions in terms of the $\alpha(k)$ sequence for a search to be purposeful will be discussed presently. Before that, however, a brief digression to the theory of stochastic orderings seems appropriate.

7. STOCHASTIC ORDERINGS

The theory of stochastic orderings deals with order relations between random variables; some results of this theory are presented here to facilitate the understanding of the proofs which follow. Theorems are not stated in their most general forms but only as general as required, and are called lemmas.

A random variable X is stochastically greater than a random variable Y, denoted by $X \geq^{st} Y$, if and only if

$$P[X > z] \geq P[Y > z]$$

for all real numbers z.

Lemma 1. Let X and Y be non-negative random variables such that $X \geq^{st} Y$. Then $EX \geq EY$.

Proof:

$$EX = \int_0^\infty P[X > z]\, dz \geq \int_0^\infty P[Y > z]\, dz = EY.$$

Lemma 2. Let $X_1 \geq^{st} Y_1$, $X_2 \geq^{st} Y_2$, and assume that (X_1,Y_1) is independent of (X_2,Y_2). Then $(X_1+X_2) \geq^{st} (Y_1+Y_2)$.

Proof: Let F_2 and G_1 be the distribution functions of X_2 and Y_1, respectively. Then

$$P[(X_1+X_2) > z] = \int_{-\infty}^{\infty} P[X_1 > z-u]\, dF_2(u)$$

$$\geq \int_{-\infty}^{\infty} P[Y_1 > z-u]\, dF_2(u)$$

$$= \int_{-\infty}^{\infty} P[X_2 > z-u]\, dG_1(u)$$

$$\geq \int_{-\infty}^{\infty} P[Y_2 > z-u]\, dG_1(u)$$

$$= P[(Y_1+Y_2) > z].$$

Lemma 3. Let X and Y be geometrically distributed random variables with success probabilities p_1 and p_2 respectively, and assume $p_1 \leq p_2$. Then $X \geq^{st} Y$.

Proof:

$$P[X > k] = (1-p_1)^k \geq (1-p_2)^k = P[Y > k].$$

Lemma 4. Let A(m) be the number of events in the area acquisition process. Then for every $m = 0,1,\ldots,$ $A(m+1) \geq^{st} A(m)$.

Proof: From equation (3) it follows that

$$A(m+1) = A(m) + X(m+1).$$

Hence

$$\begin{aligned} P[A(m+1) > k] &= P[A(m)+X(m+1) > k] \\ &= P[A(m) > k] + P[A(m)=k, X(m+1)=1] \\ &\geq P[A(m) > k]. \end{aligned}$$

8. SUFFICIENT CONDITIONS FOR PURPOSEFUL SEARCHES

The conditions a purposeful search, as defined in section 6, has to satisfy can be expressed formally as

(1) $D_{ps} \leq^{st} D_{rs}$

(2) $d_{ps}(m+1) \geq d_{ps}(m)$, $m = 1,2,\ldots,$[6]

where the subscript ps denotes a purposeful search. The following propositions provide tools to establish conditions on the $\alpha(k)$ sequence of a purposeful search.

Proposition 1. Let $\{<A_1(m), m = 0,1,\ldots>,Z\}$ and $\{<A_2(m), m = 0,1,\ldots>,Z\}$ be search processes specified by the sequences $<\alpha_1(k), k = 0,\ldots,n>$ and $<\alpha_2(k), k = 0,\ldots,n>$, respectively, and assume for $k = 0,\ldots,n$ that $\alpha_1(k) \leq \alpha_2(k)$. Then $D_1 \geq^{st} D_2$, when D_1, D_2 are the respective times-to-detection.

Proof: Let $T_1(k)$ and $T_2(k)$, $k = 1,\ldots,n$, be the inter-event times of the processes. By definition they are geometrically distributed random variables with success probabilities $\alpha_1(k-1)$ and $\alpha_2(k-1)$, respectively. Since by assumption $\alpha_1(k) \leq \alpha_2(k)$ for all $k = 0,\ldots,n$, thus $T_1(k) \geq^{st} T_2(k)$, $k = 1,\ldots,n$, by Lemma 3. The inter-event times of each process are independent by definition; hence $\sum_{i=1}^{k} T_1(i) \geq^{st} \sum_{i=1}^{k} T_2(i)$, $k = 1,\ldots,n$, by successive applications of Lemma 2. But by definition $\sum_{i=1}^{k} T(i) = S(k)$,

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[6] In the case of the systematic search: $m=1,2,\ldots,n-1$.

and thus the $k^{\underline{th}}$ event times of the processes, $S_1(k)$ and $S_2(k)$, are stochastically ordered as $S_1(k) \geq^{st} S_2(k)$, $k = 1,\ldots,n$, and hence $P[S_1(k) \leq m] \leq P[S_2(k) \leq m]$, by definition, for all real numbers m and integers $k = 1,\ldots,n$. From this it follows that $\sum_{k=1}^{n} P[S_1(k) \leq m] \leq \sum_{k=1}^{n} P[S_2(k) \leq m]$, and further, using equation (1), that $P[D_1 \leq m] \leq P[D_2 \leq m]$ for all real numbers m. Hence by definition $D_1 \geq^{st} D_2$. $\square$

Proposition 2. Let $\{\langle A(m),\ m = 0,1,\ldots\rangle, Z\}$ be a search process specified by the sequence $\langle\alpha(k),\ k = 0,\ldots,n\rangle$. Then if $\alpha(k)$ is linear in k for $k = 0,\ldots,n-1$ and $\alpha(n-1) \geq \frac{1}{n}$, the corresponding search has a non-decreasing detection rate.

Proof: From the assumption of linearity it follows that for $k = 0,\ldots,n-1$, $\alpha(k) = a-bk$ for some numbers a,b. Since for every search process $\alpha(0) = 1$, hence $a = 1$; also $b \leq \frac{1}{n}$ by the assumption $\alpha(n-1) \geq \frac{1}{n}$.

Let $\beta(x) \equiv 1-bx$, $0 \leq x \leq n-1$, $0 \leq b \leq \frac{1}{n}$; then for $k = 0,\ldots,n-1$, $\beta(k) = \alpha(k)$. Define $\theta(x) \equiv \frac{\alpha(x)}{(n-x)}$, $0 \leq x \leq n-1$; since $\theta'(x) = \frac{(1-nb)}{(n-x)^2} \geq 0$, hence $\theta(x)$ is non-decreasing in x.

In equation (6) it was shown that the detection rate d(m) can be expressed as

$$d(m) = \frac{E\alpha[A(m-1)]}{n-EA(m-1)}$$

for all integers m for which it is defined. Since in this

expression $A(m-1)$ can only take on the values $0,\ldots,n-1$, it can be re-written as

$$d(m) = \frac{E\beta[A(m-1)]}{n-EA(m-1)}$$

or, since β is a linear function, as

$$d(m) = \frac{\beta[EA(m-1)]}{n-EA(m-1)}$$

which is the same as

$$d(m) = \theta[EA(m-1)].$$

In Lemma 4 it was proved that $A(m+1) \geq^{st} A(m)$ for all non-negative integers m; thus by Lemma 1, $EA(m+1) \geq EA(m)$ for $m = 0,1,\ldots$, i.e. $EA(m)$ is increasing in m. But θ is an increasing function of its argument, and hence $d(m)$ is non-decreasing in m. $\square$

From Propositions 1 and 2 it can be concluded immediately that if the sequence $\langle\alpha(k), k = 0,\ldots,n\rangle$ of a search process is such that for $k = 0,\ldots,n-1$,

(1) $\alpha(k) \geq \alpha_{rs}(k)$,

(2) $\alpha(k)$ is linear in k,

then the corresponding search is a purposeful search.

9. FURTHER DISCUSSION OF PURPOSEFUL SEARCHES

The results of the previous section seem to indicate that any search with non-decreasing detection rate also has a time-to-detection which is stochastically less than that of a random search. That this is indeed the case can be seen from the following argument.

Consider a search with non-decreasing detection rate $d(i)$, $i = 1,2,\ldots$, and time-to-detection D. From equation (2) it follows that for any search, $d(1) = \frac{1}{n}$; hence by the assumption of non-decreasing detection rate, $d(i) \geq \frac{1}{n}$, $i = 1,2,\ldots$. But then $\prod_{i=1}^{m} [1-d(i)] \leq [1-\frac{1}{n}]^m$ for positive integers m, and thus by equations (7) and (14), $F_D(m) \geq F_{rs}(m)$. Hence $D \leq^{st} D_{rs}$.

Therefore, clause (1) in the definition of a purposeful search is redundant and can be omitted. The definition then reads:

> A purposeful search is a search whose detection rate is non-decreasing.

Clause (1) could now be added as a theorem:

> The time-to-detection of a purposeful search is stochastically less than the time-to-detection of a random search.

The sufficient conditions on the $\alpha(k)$ sequence, of course, remain unchanged.

10. EXTENSIONS OF THE SEARCH PROCESS

In this paper it was shown how a search can be expressed mathematically as a stochastic process of a special type, and how formulas for the time-to-detection and the detection rate can be derived from this process. Because of the severe limitations on the model under consideration, this effort must be regarded as an expository example only. To obtain results which are of theoretical interest and practical use it is necessary to extend the search process to more general models which take into account situations where

(1) the observer uses a detection device which is not of the definite range law type and which has a non-zero false alarm probability,

(2) the target is not uniformly distributed and/or not stationary,

(3) several observers search for a target, either independently or in a coordinated action, and

(4) the observer(s) and the target move in continuous time and space.

Some of these extensions seem to pose no difficulties. A non-zero probability of false alarms, for instance, can probably be dealt with by stochastically increasing the time between successive acquisition events. Other extensions will certainly require a great amount of further research.

In view of the structural similarities between a search in which a target is eventually detected and a piece of equipment which eventually fails, and subsequently the analogy between a time-to-detection and a time-to-failure, a detection rate and a failure rate, it is felt that this research could benefit by using some of the methods and tools developed in reliability theory.

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13. ABSTRACT

A discrete-time discrete-space search model is considered in which an observer employing an idealized detection device is searching for a uniformly distributed stationary target. The model is formulated as a discrete-time counting process, called the search process, which under weak additional conditions is uniquely determined by a sequence of probabilities. Formulas for the time-to-detection and the detection rate of a search are derived in terms of the parameters of the search process, and are applied to two special types of searches, the systematic search and the random search. Using these search types as boundary cases a purposeful search is defined, and sufficient conditions on the sequence of probabilities are established for the purposeful search. Possible extensions of the search process to less restricted models are indicated.

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